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Conference

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Conference Materials

Content

into digital channels. Take the example of two leading banks in Georgia: the retail offloading ratio of TBC Bank was 97% in 2021 [6], while the share of retail transactions through digital channels increased car ownerships in the USA, which increased the mobility of shoppers. With the increased mobility, it became a lifestyle to combine shopping, entertainment and leisure in one trip as it was convenient for consumers. The evolved lifestyle has manifested itself in the changed consumer behavior. The retailing sector responded to this change with the emergence of shopping malls by means of providing all services in one place in a manner convenient for consumers.

Now we are in the position to answer the main question posed in this article: *does the value proposition of the financial services providers (including Neobanks) match the changing purchasing habits of Consumers?* If we agree that the value for a modern-day consumer is "convenience" by means of receiving virtually all services in one place (i.e. in a mobile phone), then the financial service providers should respond by creating the respective ecosystems. Is this the case?

The answer is No. Today, the financial services industry is fragmented into digital payments, loyalty platforms and neobanking/digital banking services to mention just a few. In addition, there are numerous companies offering E-commerce services/marketplaces, which increase the degree of defragmentation from the consumers' perspective. All these services are facing some problems when

Fig. 5 Comparison of loyalty platforms

The resulting ecosystem is shown in Fig.6. The business logic behind such ecosystem is the following: digital payments functionality will generate a large base of participating companies, as these companies look for reducing the transaction fees. Loyalty platforms will generate a large customer base and provide customer behavioral information to the participating banks (with the consent of customers). Banks will score the customer behavior information and will embed their offerings into the marketplace. As a result the banks will be able to issue online loans for consumers shopping at marketplace. The customers of the ecosystem will receive all the above mentioned services in one place (convenience). The resulting ecosystem will be profitable due to high CLV of banking products and services.

Summary

In order to summarize the future development of neobanking in just a few words, it can be stated: neobanking is the future. A lot of new financial ecosystems will emerge, some of them in partnership with the existing banking institutions. The examples of such initiatives can be seen everywhere: Visa and Mastercard are becoming digital, Apple and Amazon are incorporating the financial services, Paypal is extending its services into the loyalty industry, even Twitter and Google are seeing themselves as payment service providers. It only remains to see how these neobanking ecosystems will be reshaped in the coming decade.

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The importance of venture capital in innovative investment projects

Dali Magrakvelidze
Georgian Technical University
E-mail: d.magrakvelidze@gtu.ge

Abstract

Innovation makes it possible to produce more products with less materials and resources, which in turn leads to economic growth. The main problem of financing innovative projects is the high risk of returns and long payback. Most of these projects do not have enough guarantee funds, their resources are limited, and only their own ideas and technologies are the backbone. Due to the high risk of innovative projects, it is necessary to use venture capital to finance them.

Key words: Venture capital, venture business, innovation, "valley of death".

* * * * *

In the modern world, as the population grows, the role of technological innovations in meeting human needs increases, as they change the world economy and contribute to the economic growth of countries.

Innovation and entrepreneurship are the kernels of a capitalist economy. New businesses, however, are often highly-risky and cost-intensive ventures. As a result, external capital is often sought to spread the risk of failure. In return for taking on this risk through investment, investors in new companies are able to obtain equity and voting rights for cents on the potential dollar. Venture capital, therefore, allows startups to get off the ground and founders to fulfill their vision.

Venture business involves financing new ideas, progressive scientific and technical developments and bringing them down to a suitable level for sale, i.e. commercialization. Venture business requires a lot of knowledge, a lot of money, and a lot of guts, but if successful, it can be hugely profitable. This type of business does not actually exist in our country, because we do not have the experience of working with new technologies and risky investments, as well as the financial infrastructure.

Venture capital (VC) is a form of private equity and a type of financing that investors provide to startup companies and small businesses that are believed to have long-term growth potential. Venture capital generally comes from well-off investors, investment banks, and any other financial institutions.

However, it does not always take a monetary form; it can also be provided in the form of technical or managerial expertise. Venture capital is typically allocated to small companies with exceptional growth potential, or to companies that have grown quickly and appear poised to continue to expand. Venture capital funds manage pooled investments in high-growth opportunities in startups and other early-stage firms and are typically only open to accredited investors.

One important difference between venture capital and other private equity deals, however, is that venture capital tends to focus on emerging companies seeking substantial funds for the first time, while private equity tends to fund larger, more established companies that are seeking an equity infusion or a chance for company founders to transfer some of their ownership stakes.

Venture capital is a subset of private equity (PE). While the roots of PE can be traced back to the 19th century, venture capital only developed as an industry after the Second World War.

Harvard Business School professor Georges Doriot is generally considered the "Father of Venture Capital." He started the American Research and Development Corporation (ARD) in 1946 and raised a $3.5 million fund to invest in companies that commercialized technologies developed during WWII. ARDC's first investment was in a company that had ambitions to use x-ray technology for cancer treatment. The $200,000 that Doriot invested turned into $1.8 million when the company went public in 1955. [3]

Business expertise. Aside from the financial backing obtaining venture capital financing can a startdeath" is the different goals of investors and businessmen (developers), the former strive for quick profit, and the latter are focused on obtaining scientific results.

For small businesses, or for up-and-coming businesses in emerging industries, venture capital is generally provided by high net worth individuals (HNWIs)—also often known as “angel investors"—and venture capital firms. The National Venture Capital Association (NVCA) is an organization composed of hundreds of venture capital firms that offer to fund innovative enterprises.

Common occurrence among angel investors is co-investing, in which one angel investor funds a venture alongside a trusted friend or associate, often another angel investor.

While both provide money to startup companies, venture capitalists are typically professional investors who invest in a broad portfolio of new companies and provide hands-on guidance and leverage their professional networks to help the new firm. Angel investors, on the other hand, tend to be wealthy individuals who like to invest in new companies more as a hobby or side-project and may not provide the same expert guidance. Angel investors also tend to invest first and are later followed by VCs.

Due to the industry's proximity to Silicon Valley, the overwhelming majority of deals financed by venture capitalists are in the technology industry—the internet, healthcare, computer hardware and services, and mobile and telecommunications. But other industries have also benefited from VC funding.

Venture capital is also no longer the preserve of elite firms. Institutional investors and established companies have also entered the fray. For example, tech behemoths Google and Intel have separate venture funds to invest in em50(7r1(o)8(g)-3(l)6(e)-5g5(h)-8(a)3(v)8(e)-30(sepa)3(ra)6 on the pot

Modern theories of leadership and types of leaders

Ketevan Kiguradze

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based on the works of (Bass, B. M., & Riggio, R. E. ., 2006) (Burton Nanus, Warren G. Bennis, 2006) made important contributions to the development of the theory. According to (Bass, B. M., & Riggio, R. E. ., 2006), the popularity of transformational theory is likely due to its emphasis on intrinsic motivation and follower development. According to this theory, people at the level of change and uncertainty need inspiration and faith

According to House's charismatic leadership theory, its face-to-face outcome is the follower's trust in the leader's ideology. Recognition of the leader without any doubts or questions.

Authentic leadership is one of the newest areas in leadership research. The theory focuses on how "real" and how authentic leadership is. There are several definitions of authentic leadership that explain it from different perspectives, they are: intrapersonal - processes taking place inside the leader's personality, self-knowledge, self-regulation, and self-evaluation; Developing - leadership behavior that is formed from the positive psychological characteristics and high quality of the leader. This is what is formed in people throughout life. Interpersonal - is built on relationships and involves achieving interactions between leaders and followers. It is a two-way process, as leaders influence followers and vice versa.

Today, one of the most recognized approaches in the field of leadership research is (House). The theory of conformity of means and ends. The essence of this theory lies in what the leader does to motivate subordinates to achieve the group and organization's goal. 1. Effective leaders clearly define the goals that subordinates are trying to achieve by working; 2. They reward subordinates according to the work done and the goal achieved and 3. They make clear the path that leads to the work goal. According to this theory, the steps a leader should take to motivate subordinates depend on both the subordinates and the type of work performed. In the theory of compatibility of the goal and the means, four behaviors of the leader are distinguished: 1. directive behavior; 2. Supportive behaviors; 3. complicity behavior; 4. Achievement-oriented behavior. Therefore, leaders must decide for themselves which behavior to use during the task to be performed by the subordinate in order to motivate them to perform the task.

Leadership concepts address the factors that leaders consider when applying leadership styles and overseeing an individual team. These principles focus on the ideas and perceptions about the qualities that leaders should have and how they should perform in the role of leader. In addition, leadership concepts help professionals understand what kind of skills and character traits they need to develop to advance in leadership roles.

The concepts of leadership differ from leadership theories in several ways. For example, is the pts of bordd hed.0612 02 regs

According to the studies by (Kirkpatrick, S.A. and Locke, E.A, 1991)have identified six traits that distinguish leaders from others. These are: Attitude, motivation, honesty, self-confidence, cognitive abilities and knowledge of the case. They think people with similar traits can be born or acquired over a lifetime They are. These 6 traits are exactly the traits that leaders need. These qualities of a leader distinguish people from each other and therefore, these differences are an important part of the leadership process. Also, empirical research (Peter G Northouse, 2010) conducted in the 1990 diversity. Brave - have the patience to achieve the goal, despite seemingly insurmountable obstacles. Exercise self-confidence in times of stress. Direct - Use common sense to make the right decisions at the right time. Imaginative - Make timely and appropriate changes in your thinking, plans and methods. Show creativity by thinking of new and better goals, ideas and problems. (John Whitehead, 2016)

Leadership theories study the qualities of effective leaders, including the qualities of effective and influential leaders, patterns of behavior, and actions. Leadership theories focus on explaining what makes good leaders by focusing on different behaviors and qualities that professionals can develop to become good leaders. While the concepts of leadership are qualities in themselves, leadership theories are the study and explanation of these qualities and their impact ondc1(p)54?/F1 10dlv

How to Compute the Gradient of the Analytically Unknown Value Function

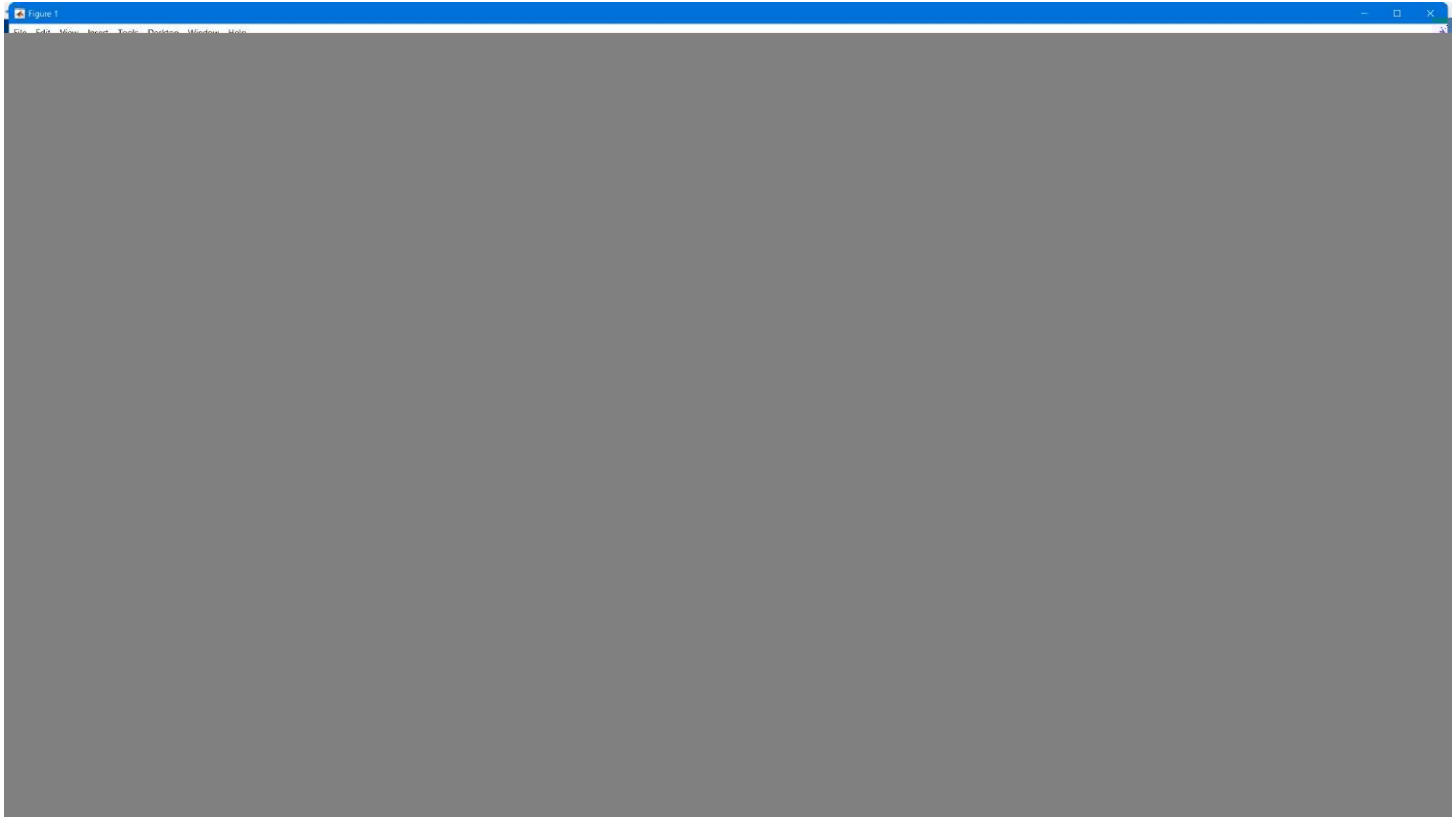
Section 1. The Basic Idea of the Research Project

It is well known that vast majority of the real-world optimization problems cannot be solved analytically in closed form since they are highly nonlinear by their intrinsic nature.

Denote ( )

The ( ) of the optimization problem is often convex (or semi convex) in multidimensional argument (for example, in engineering thermodynamics it is seems intuitively reasonable, bu6rTn.5innn ui1ennieliietі6(onse).(,)10(4(T(t)h(n)e(n) ea)] TJ

Section 2. Convex Envelope Animations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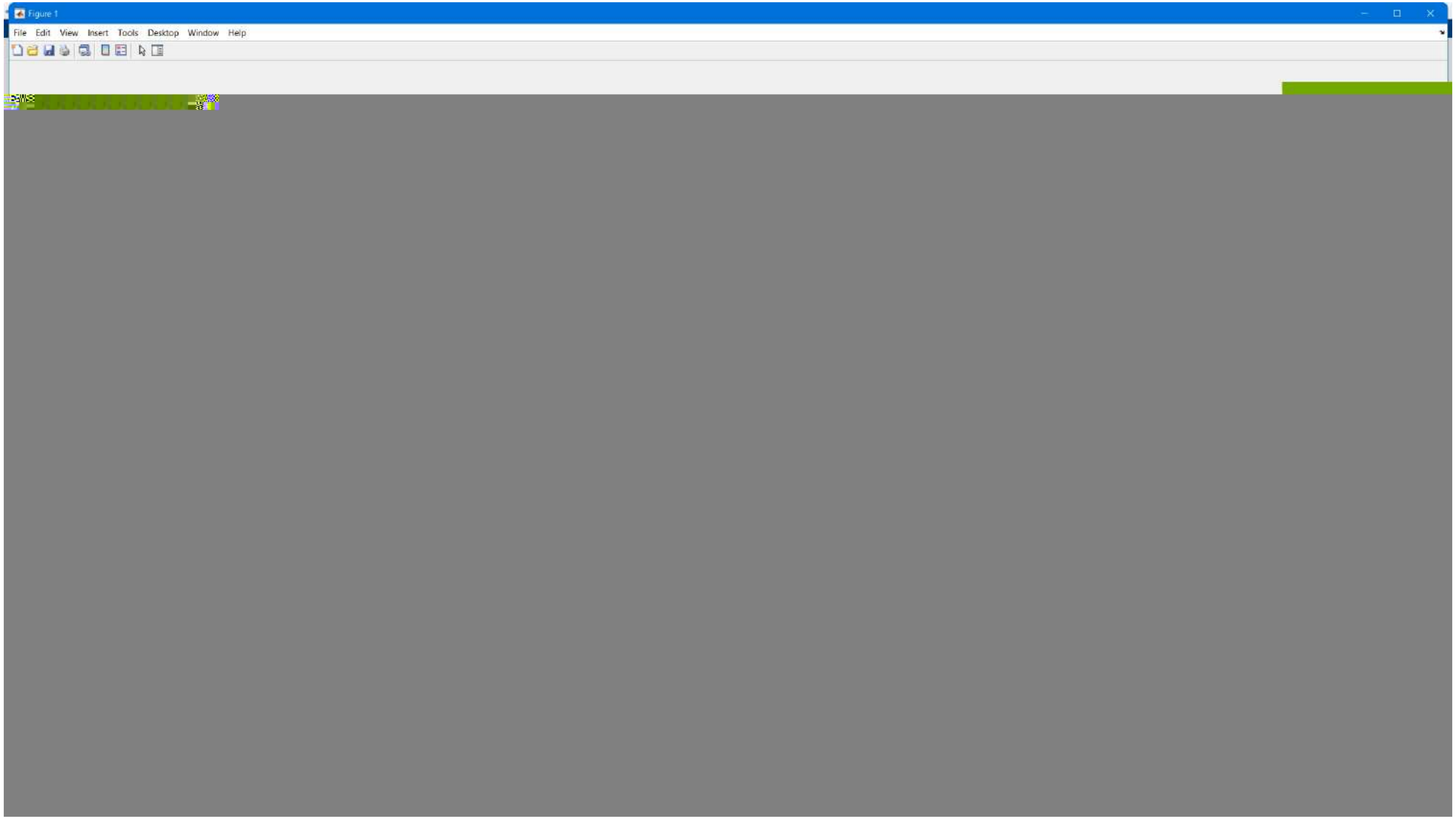
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Section 3. The -Approximation of the Gradient of the Semiconvex Function through the Convex Envelope

Let : →

Taking successively = 0 and = 0 in (3.3) we get

$$\| \quad (\ )\|_{\infty(\ )} \le \| \quad \|_{\infty(\ )}, \qquad\qquad (3.4)$$
$$\| \quad (\ )\|_{\infty(\ )} \le \| \quad \|_{\infty(\ )}.$$

$(\ ) \cap {}^{\infty}(\ ) \quad \to \quad (\ )$ *possesses the following important property*

$$| \quad (\ ) - \quad (\ )|^2 \cdot \frac{2}{\quad} \le 5 \quad \cdot \| \quad - \quad \|_{\infty(\ )} \big(\| \quad \|_{\infty(\ )} + \| \quad \|_{\infty(\ )} \big). \qquad (3.5)$$

We have from the bound (3.4) that the convex functions

Consider now the bounded viscosity solution of the equation (3.1) which is assumed to be semiconvex with semiconvexity constant ≥ 0 and its uniform continuous numerical approximation , i.e.

$$\|\ \ -\ \ \|_{\infty(\ )} \underset{0}{\longrightarrow} 0. \tag{3.7}$$

Further consider the bounded continuous functions

$$+\ \ \cdot\ \ _0 \ \text{ and } \ +\ \ \cdot\ \ _0 \tag{3.8}$$

and their convex envelopes

$$\text{conv}(\ \ +\ \ \cdot\ \ _0) \ \text{ and } \ (\ \ +\ \ \cdot\ \ _0), \tag{3.9}$$

where

$$_0(\ ) = \frac{1}{2} \cdot |\ \ |^2.$$

The next proposition is the main result of Section 3.

Section 4. Computation of the Gradient of the Solution of Monge-Ampere Partial Differential Equation in a Planar Domain

We discuss next the Monge-Ampere equation. The Monge-Ampere equation is a fully nonlinear elliptic PDE. Applications of the Monge-Ampere equation appear in the classical problem of prescribed Gauss curvature and in the problem of optimal mass transportation (with quadratic cost).

We shall present a simple (nine point stencil) finite difference method which performs well for smooth as well as for singular solutions. The Monge-Ampere PDE in a planar domain ^{2} is the following

$$\Big(\quad (\;)\Big) = \; (\;), \; (\;) \geq 0,$$

or equivalently

$$\frac{2}{\;\;2} \cdot \frac{2}{\;\;2} - \left(\frac{2}{\quad}\right)^2 = \quad \text{with Dirichlet boundary conditions} \quad = \quad \text{on} \tag{4.1}$$

and the additional convexity constraint

$$(\; , \;) \;\; \text{is convex in} \;\; , \tag{4.2}$$

which is required for the equation to be elliptic. Without the convexity constraint this equation does not have a unique solution. For example, taking the boundary function $=$

Now solving for and selecting the smaller one (in order to select the locally convex solution), we obtain

$$= \frac{1}{2}({}_1 + {}_2) - \frac{1}{2}\sqrt{({}_1 - {}_2)^2 + \frac{1}{4}({}_3 - {}_4)^2 + {}^4} \;. \qquad (4.7)$$

We can now use Gauss-Seidel iteration to find the fixed point of (4.7).

The Dirichlet boundary conditions are enforced at boundary grid points. The convexity constraint (4.2) is not enforced (beyond the selection of the positive root in (4.7).

Next we consider two exact solutions for the Monge-Ampere PDE (4.1), (4.2) on the square [0,1] × [0,1].

Figure 1

Figure

The Monge-Ampere equations (the Examples 4.1 and 4.2) are considered on the square [0,1] × [0,1].

In the tables below for the different grid points we compute the number of iterations, the computation times, the errors of approximation of the exact solution and of the exact gradient.

Computation times and errors for the exact solution and its gradient for the Example 4.1 on an × grid:

21	1362	1 sec.	1.5×10^{-4}	0.1255	0.011
61	10840	10 sec.	1.8×10^{-5}	0.0441	0.0038
101	28764	60 sec.	6.7×10^{-6}	0.0267	0.0023
141	54802	300 sec.	3.4×10^{-6}	0.0192	0.0016

We give the surface plots (for Examples 4.1 and 4.2) of the following functions:

a) the exact solution,

b) finite difference numerical approximation,

c) the convex envelope of the numerical approximation,

d) partial derivative w.r. to of the exact solution,

e) partial derivative w.r. to of the exact solution,

f) partial derivative w.r. to of the convex envelope,

g) Partial derivative w.r. to of the convex envelope.

Section 5. Pricing and Hedging of American Options written on Multiple Assets

In this section we study the multidimensional parabolic obstacle problem and its relation to the pricing and hedging of American options written on multiple assets. We shall consider the so called strong solutions of parabolic obstacle problem that have been studied, for example, in Friedman [3, Chapter 1]. Strong solutions have second order Sobolev (weak) derivatives so that the Partial Differential Equation (PDE) can be written pointwisely a.e., strong solutions should be preferable in financial applications because of their better regularity properties.

The above obstacle problem appears naturally in the valuation of American type Claims in financial market. The obstacle is the so called payoff function and the solution of the obstacle problem is the value function of the American option written on multiple assets. A good background study is given in the paper by Broadie and Detemple [1].

The semiconvexity is a natural property of a large class of value functions of the optimization problems (see, for instance, Cannarsa and

American option can be exercised by its holder (as an opposite to European option) at any time up to and including expiry. This makes their pricing mathematically challenging and few closed form solutions have been found. American options are important because they are very widely traded. At least as important as the pricing of American options are the hedging issues that are crucial for the writer of the option.

In this section we study the parabolic obstacle problem in the strong sense. More precisely, we seek a solution (,), which belongs to the parabolic Sobolev space (see, for example, Krylov [6, Chapter 2]) and satisfies a system of inequalities

$$(\, , \,) \leq 0, \qquad (\, , \,) \geq (\,), \qquad (\, , \,) \cdot \big((\, , \,) - (\,) \big) = 0 \tag{5.1}$$

(×) with terminal condition

$$(\, , \,) = (\,), \tag{5.2}$$

where ( ),

when the obstacle ( ) is non-smooth there are not many known techniques to be used in the study of the obstacle problem. Our objective is to develop some new results for the nonsmooth case, with focus on applications to American type options written on multiple assets, which is an active research area at present in mathematical finance.

We will consider the pricing and hedging of multidimensional American options in a financial market driven by a general multidimensional Ito diffusion. The American option is a financial contract, assuming a time horizon of > 0 and a market consisting of assets $( ) = \big({}_1( ), \dots , ( )\big)$ giving a payoff at time equal to $\Psi\big({}_1( ), \dots , ( )\big)$ where Ψ

We assume that there exists a risk-neutral martingale measure , such that with respect to the logarithms of the prices $( ) = \left(\ln({}_1( )), \ldots, \ln( )\right)$

We will assume that the operator is uniformly parabolic in the sense that there exists > 0 such that

$$(\ , \ ) \cdot \quad \cdot \quad \geq \quad \cdot | \ |^2, \qquad \text{whenever} \ (\ , \ ) \qquad \times$$

$_{, \ =1}$

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The convexity (semiconvexity) of the value function (,) of the American option for arbitrary fixed time instant is the crucial point for our new device of the construction of the nearly optimal discrete time delta hedging strategies for American options written on multiple assets.

Indeed recently in the paper by Shashiashvili

But the perfect hedging in continuous time requires the continuous rebalancing of the writer's portfolio in the underlying assets and the money market account, which is impossible in practice. In reality, the writer trades only at some discrete instants of time at which he rebalances his portfolio. Moreover, the delta-hedging requires the knowledge of the gradient (,) of the value function (,), but the explicit form neither of the value function, nor of its partial derivatives is known even in the simplest Black-Sholes model for American put option with finite horizon > 0.

Several approximation methods were devised in order to compute the value function of the American option. In particular, finite difference methods were developed in Wilmott, Dewynne, and Howison [10], and Jaillet, Lamberton, and Lapeyre [4]. We assume here that we have already been given some continuous in the argument uniform approximation (,) to the unknown value function

Let ( ) and ( ) be two semiconvex functions in with the semiconvexity constants and , respectively (see Cannarsa and Sinestrari [2, Chapter 1, Definition 1.1.1]) and ( ) be a nonnegative

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Thank you very much for your attention!

The Adomian series representation of some quadratic BSDEs

R. Tevzadze

A . The representation of the solution of some Backward Stochastic Di erential Equation as an in nite series is obtained. Some exactly solvable examples are considered.

2020 Mathematics Subject Classi cation. 90A09, 60H30, 90C39

Keywords: Stochastic exponential, martingale, Adomian series, Brownian Motion.

1 Introduction

In a number of papers[1,2] Adomian develops a numerical technique using special kinds of polynomials for solving non-linear functional equations. However, Adomian and his collaborators did not develop widely the problem of convergence.

In this article we will study by Adomian technique some kind of quadratic backward martingale equation and prove the convergence of the series. For example we tackle an equation of the form

$$E_T(m)E_T^{\alpha}(m^?) = c\exp f \quad g \tag{1}$$

w.r.t. stochastic integrals $m = \int f_s dW_s;\ m^? = \int g_s dW_s^?$ and real number c, where $(W; W^?)$ is 2-dimension Brownian Motion and is a random variable.

Equations of such type are arising in mathematical nance and they are used to characterize optimal martingale measures (see, Biaginiat at al (2000), Mania and Tevzadze (2000), (2003),(2006)). Note that equation (1) can be applied also to the nancial market models with in nitely many assets (see M. De Donno at al (2003)). In Biagini at al (2000) an exponential equation of the form

$$\frac{E_T(m)}{E_T(m^?)} = ce^{\int_0^{\hat{}} \lambda^2 ds}$$

was considered (which corresponds to the case = 1).

Our goal is to show the solvability of the equation (1) using the Adomian method proving the convergence of series. On the one hand, a simpler proof of solvability is obtained. On the other hand, it allows to obtain the approximation of the solution. It is possible to nd a solution in the form of series, if we de ne a sequence of martingales w.r.t. the measure $E_T(\sum_i^n m_i + \sum_i^n m_i^?)$ P from equations $c^0 E_T(m^0_{n+1} + m^{0?}_{n+1}) = E_T^2(m_n^{0?})$, where m^0_{n+1} =

where   is a given F_T-measurable random variable and   is a given real number. A solution of equation (2) is a triple $(c; m; m^\perp)$, where c is strictly positive constant, $m \in M$ and $m^\perp \in M^\perp$. Here $\mathcal{E}(X)$ is the Doleans-Dade exponential of X.

It is evident that if   = 1 then equation (2) admits an "explicit" solution. E.g., if   = 1 and   is bounded, then using the unique decomposition of the martingale $E(\exp\{\ \}/F_t)$

$$E(\exp\{\ \}/F_t) = E\exp\{\ \} + m_t(\ ) + m_t^\perp(\ ); \quad m(\ ) \in M; \quad m^\perp(\ ) \in M^\perp; \tag{3}$$

it is easy to verify that the triple $c = \frac{1}{E\exp\{\eta\}}$,

$$m_t = \int_0^t \frac{1}{E(\exp\{\ \}/F_s)} dm_s(\ ); \quad m_t^\perp = \int_0^t \frac{1}{E(\exp\{\ \}/F_s)} dm_s^\perp(\ )$$

satisfies equation (2).

Our aim is to prove the existence of a unique solution of equation (2) for arbitrary $\neq 0$ and   of a general structure, assuming that it satisfies the following boundedness condition:

B)   is an F_T-measurable random variable of the form

$$= \ + A_T; \tag{4}$$

where $\in L^\infty$,   is a constant and $A = (A_t; t \in [0;T])$ is a continuous F-adapted process of finite variation such that

$$E(var_T(A) - var_\tau(A)/F_\tau) \le C$$

for all stopping times   for a constant $C > 0$.

One can show that equation (2) is equivalent to the following semimartingale backward equation with the square generator

$$Y_t = Y_0 - \frac{}{2}A_t - \langle L \rangle_t - \frac{1}{}\langle L^\perp \rangle_t + L_t + L_t^\perp; \quad Y_T = \frac{1}{2} \ . \tag{5}$$

We use also the equivalent equation of the form

$$L_T + L_T^\perp = c + \langle L \rangle_T + \frac{1}{}\langle L^\perp \rangle_T + \frac{}{2}A_T.$$

w.r.t. $(c; L; L^\perp)$.

We use notations $jMj_{\mathrm{BMO}} = \inf fC : E^{1}$

In particular, if $\|A\|_\omega < \infty$ then the martingale $E(A_T|F_t)$ belongs to the BMO space and

$$\|E(A_T|F_\cdot)\|_{\mathrm{BMO}} \le \|A\|_\omega.$$

Proof. By the Ito formula

$$Y_t^2 = 2\int_0^t Y_s dm_s + 2\int_0^t Y_s dA_s + \langle m\rangle_t.$$

Assume that inequality (8) is valid for any $k \le n$ and let us show that

$$|L^{(n+1)} + L^{\star(n+1)}|_{\mathrm{BMO}} \le a_{n+1}(1 + |\ \ |)^{n+1}|L^{(0)} + L^{\star(0)}|^{n+2}_{\mathrm{BMO}}. \qquad (9)$$

Applying Lemma 1 for $Y_t^{(n+1)}$ and the Kunita-Watanabe inequality we have

$$|L^{(n+1)} + L^{\star(n+1)}|_{\mathrm{BMO}}$$

$$\le \operatorname*{ess\,sup}_{\tau} \sum_{k=0}^{n} E(var_\tau^T(\sum_k^n \langle L^{(k)}, L^{(n-k)}\rangle + \ \ \langle L^{\star(k)}, L^{\star(n-k)}\rangle)|F_\tau)$$

$$\le \sum_{k=0}^{n} \operatorname*{ess\,sup}_{\tau} E^{\frac12}(var_\tau^T\langle L^{(k)}\rangle|F_\tau)E^{\frac12}(var_\tau^T\langle L^{\star(n-k)}\rangle|F_\tau)$$

$$+|\ \ |\sum_{k=0}^{n} \operatorname*{ess\,sup}_{\tau} E^{\frac12}(var_\tau^T\langle L^{\star(k)}\rangle|F_\tau)E^{\frac12}(var_\tau^T\langle L^{\star(n-k)}\rangle|F_\tau)$$

$$\le \sum_{k}^{n} |L^{(k)}|_{\mathrm{BMO}}|L^{(n-k)}|_{\mathrm{BMO}} + |\ \ ||L^{\star(k)}|_{\mathrm{BMO}}|L^{\star(n-k)}|_{\mathrm{BMO}}$$

$$\le (1 + |\ \ |)\sum_{k=0}^{n} |L^{(k)} + L^{\star(k)}|_{\mathrm{BMO}}|L^{(n-k)} + L^{\star(n-k)}|_{\mathrm{BMO}}. \qquad (10)$$

Therefore, from (10), using inequalities (8) for any $k \le n$, we obtain

$$|L^{(n+1)} + L^{\star(n+1)}|_{\mathrm{BMO}}$$

$$\le (1+|\ \ |)\sum_{k=0}^{n} a_k(1+|\ \ |)^k|L^{(0)}+L^{\star(0)}|^{k+1}_{\mathrm{BMO}}a_{n-k}(1+|\ \ |)^{n-k}||L^{(n-k)}+L^{\star(n-k)}|^{n-k+1}_{\mathrm{BMO}}$$

$$\le (1 + |\ \ |)^{n+1}|L^{(0)} + L^{\star(0)}|^{n+2}_{\mathrm{BMO}} \sum_{k=0}^{n} a_k a_{n-k} =$$

$$= a_{n+1}(1 + |\ \ |)^{n+1}|L^{(0)} + L^{\star(0)}|^{n+2}_{\mathrm{BMO}}$$

and the validity of inequality (8) follows by induction.

Theorem 1. *The series* $\sum_{n \ge 0}(L^{(n)} + L^{\star(n)})$ *is convergent in BMO-space, if and* $|\ \ |_1$ *are small enough and the sum of series is a solution of the equation (5).*

Proof. Without loss of generality assume that

We have

$$L_T^{(0)} + L_T^{(0)?} = c_0 + \int_0^T (T \quad s) W_s dW_s + \int_0^T (T \quad s) W_s^? dW_s^?;$$

$$L_T^{n+1)} + L_T^{(n+1)?} = c_n + \sum_{k=0}^{n} hL^{(k)}$$

Introducing $(s) = \sum_{n=0}^{1} {}_n s^{2n+1}; \quad (s) = \sum_{n=0}^{1} {}_n s^{2n+1}$ one obtains

$$ {}^0(s) = {}_0 + \sum_{n=0}^{1}(2n+3)\ {}_{n+1}s^{2n+2} $$

$$ = 1 + 2\sum_{n=0}^{1}\sum_{k=0}^{n}(\ {}_k\ {}_n\ {}_k)s^{2n+2} = 1 + 2a^2(s); $$

$$ {}^0(s) = {}_0 + \sum_{n=0}^{1}(2n+3)\ {}_{n+1}s^{2n+2} $$

$$ = 1 \quad 2\sum_{n=0}^{1}\sum_{k=0}^{n}\ {}_k\ {}_n\ {}_k s^{2n+2} = 1 \quad 2\ {}^2(s): $$

I.e.

$$ {}^0(s) = 1 + 2a^2(s); \quad (0) = 0; \qquad (11) $$
$$ {}^0(s) = 1 \quad 2\ {}^2(s); \quad (0) = 0: $$

Thus

$$ (s) = \frac{1}{p\overline{2}}\tan(p\overline{2}s); \quad (s) = \frac{1}{p\overline{2}}\tanh(p\overline{2}s): $$

If $T < \frac{\pi}{2p\overline{2}}$

We have

$$L_T^{(0)} = EL_T^{(0)} + \int_0^T (T \quad s) W_s^{?} dW_s; \; L_T^{(0),?} = EL_T^{(0),?} + \int_0^T (T \quad s) W_s dW_s^{?};$$

$$L_T^{(n+1)} +$$

Using representation of integrands by stochastic derivatives we get

$$(T \quad t)^{2n+3}(\ _{n+1}W_t + \ _{n+1}W_t^?)$$
$$= E[D_t(\sum_{k=0}^{n} hL^{(k)}; L^{(n \ k)} i_T \quad \sum_{k=0}^{n} hL^{(k)?}; L^{(n \ k)?} i_T)jF_t]$$
$$= 2\sum_{k=0}^{n}[(\ _k \ _n \ _k \quad \ _k \ _n \ _k)W_t + (\ _k \ _n \ _k + \ _k \ _n \ _k)W_t^?]\int_t^T (T \quad s)^{2n+2} ds$$
$$= \frac{2(T \quad t)^{2n+3}}{2n+3}\sum_{k=0}^{n}[(\ _k \ _n \ _k \quad \ _k \ _n \ _k)W_t + (\ _k \ _n \ _k + \ _k \ _n \ _k)W_t^?];$$

$$(T \quad t)^{2n+3}(\ _{n+1}W_t \quad \ _{n+1}W_t^?)$$
$$= E[D_t^?(\sum_{k=0}^{n} hL^{(k)}; L^{(n \ k)} i_T \quad \sum_{k=0}^{n} hL^{(k)?}; L^{(n \ k)?} i_T)jF_t]$$
$$= 2\sum_{k=0}^{n}[\quad (\ _k \ _n \ _k \quad \ _k \ _n \ _k)W_t^? + (\ _k \ _n \ _k + \ _k \ _n \ _k)W_t]\int_t^T (T \quad s)^{2n+2} ds$$
$$= \frac{2(T \quad t)^{2n+3}}{2n+3}\sum_{k=0}^{n}[\quad (\ _k \ _n \ _k \quad \ _k \ _n \ _k)W_t^? + (\ _k \ _n \ _k + \ _k \ _n \ _k)W_t]:$$

Equalising coe cients at $W; W^?$ we obtain the desired formula. One can be checked that $\lim_{n!1} \sqrt[n]{ja_nj} = 0; \lim_{n!1} \sqrt[n]{jb_nj} = 0:$ Introducing $(s) = \sum_{n=0}^{1} \ _n s^{2n+1}; \quad (s) = \sum_{n=0}^{1} \ _n s^{2n+1}$ one obtains

$$L_t = L_0 + \int_0^t (\ (T \quad s)W_s + \ (T \quad s)W_s^?)dW_s;$$
$$L_t^? = L_0^? + \int_0^t (\ (T \quad s)W_s \quad (T \quad s)W_s^?)dW_s^?:$$

On the other hand we can derive ODE for the pair (;)

$$^0(s) = 2 \ ^2(s) \quad 2 \ ^2(s); \quad (0) = 0; \tag{12}$$
$$^0(s) = 1 + 4 \ (s) \ (s); \quad (0) = 0:$$

Indeed

$$^{0}(s) = {}_{0} + \sum_{n=0}^{1}(2n+3)\;{}_{n+1}s^{2n+2}$$

$$= 2\sum_{n=0}^{1}\sum_{k=0}^{n}(\;{}_{k}\;{}_{n}\;{}_{k}\qquad{}_{k}\;{}_{n}\;{}_{k})s^{2n+2}\text{n}$$

Proposition 3.

$$Ee^{\int_0^{T} W^2 dt} = \begin{cases} \frac{1}{\sqrt{\cos(\sqrt{2}T)}} \end{cases}$$

If $T > \frac{\pi}{2^p \overline{2}}$ then $Ee^{\int_0 \; W^2 dt} > Ee^{\int_0^{\frac{\pi}{2\sqrt{2}}} W^2 dt} = 1:$

Lemma 3. *Let* $(a_n)_{n \quad 0}$ *be a solution of the system*

$$a_0 = 1; \; a_{n+1} = \sum_{k=0}^{n} a_k a_{n \quad k}: \tag{13}$$

Then $a_n = \frac{1}{4n+2}\binom{2n+2}{n+1}$.

Proof. For the series $u(\;) = \sum_{n=0}^{1} a_n \quad ^n$ from (13) we get equation $u(\;) = 1 + \quad u^2(\;)$, with the roots $u(\;) = \frac{1}{2\lambda}(1 \quad ^p\overline{1 \quad 4\;})$. The equality $u(\;) = \frac{1}{2\lambda}(1 + \; ^p\overline{1 \quad 4\;})$ is impossible, since decomposition of the right hand side is starting from the term $\frac{1}{\lambda}$. Therefore, equality $a_n = \frac{1}{4n+2}\binom{2n+2}{n+1}$ follows from the Taylor expansion of $1 \quad ^p\overline{1 \quad 4\;}$, since

$$u(\;) = \frac{1}{2}(1 \quad ^p\overline{1 \quad 4\;})$$

$$= \frac{1}{2}\sum_{n \quad 1} \frac{\frac{1}{2}(\frac{1}{2} \quad 1) \quad (\frac{1}{2} \quad n+1)}{n!}(\quad 4)^n \quad ^{n \quad 1}$$

$$= \frac{1}{2}\sum_{n \quad 1} \frac{(2 \quad 1) \quad (2n \quad 2 \quad 1)}{2^n n!} 4^n \quad ^{n \quad 1}$$

$$= \frac{1}{2}\sum_{n \quad 1} \frac{(2n \quad 3)!!}{n!} 2^n \quad ^{n \quad 1} = \frac{1}{2}\sum_{n \quad 1} \frac{1}{2n \quad 1}\binom{2n}{n} \quad ^{n \quad 1}:$$

Lemma 4. *There exist sequences* $(m_i; i \quad 1) \; 2 \; M; \; (m_i^?; i \quad 1) \; 2 \; M^?;$ *such that* $e^\eta = c_1 \frac{E_{\tilde{}}\,(m_1)}{E_{\tilde{}}\,(m_1^\perp)} E_T^2(m_1^?)$ *and*

$$e^\eta = c_n \frac{E_T(\sum_i^n m_i)}{E_T(\sum_i^n m_i^?)} E_T^2(m_n^{\prime ?}); \; n \quad 2; \tag{14}$$

where $m_n^{\prime ?} = m_n^? \quad \langle m_n^?; \sum_i^{n \quad 1} m_i^? \rangle$.

Proof. The theorem will be proved by induction. Assume (14) is valid for n. There exist such martingales $m_{n+1}; m_{n+1}^?$ that $c^0 E_T(m_{n+1}^0 + m_{n+1}^{0?}) = E_T^2(m_n^{\prime ?})$ and

$$m_{n+1}^0 = m_{n+1} \quad \langle m_{n+1}; \sum_i^n; m_i \rangle; \;\; m_{n+1}^{0?} = m_{n+1}^? \quad \langle m_{n+1}^?; \sum_i^n m_i^? \rangle$$

are martingales w.r.t. $E(\sum_i^n m_i + m_i^?)$ $P:$ Thus

$$e^{\eta} = c_n c^0 \frac{E_T(\sum_i^n m_i)}{E_T(\sum_i^n m_i^?)} E(m_{n+1} \quad hm_{n+1}; \sum_i^n m_i i + m_{n+1}^? \quad hm_{n+1}^?; \sum_i^n m_i^? i)$$

$$= c_{n+1} \frac{E_T(\sum_i^n m_i) E_T(m_{n+1} \quad hm_{n+1}; \sum_i^n m_i i)}{E_T(\sum_i^n m_i^?) E_T(m_{n+1}^? \quad hm_{n+1}^?; \sum_i^n m_i^? i)} E_T^2(m_{n+1}^? \quad hm_{n+1}^?; \sum_i^n m_i^? i)$$

$$= c_{n+1} \frac{E_T(\sum_i^{n+1} m_i)}{E_T(\sum_i^{n+1} m_i^?)} E_T^2(m_{n+1}'^?):$$

Remark. If we will prove the convergence of series $\sum_i m_i; \sum_i m_i^?$, then $m_n^?$! $0; m_n'^?$! $0;$ $E(m_n'^?)$! 1 and $e^{\eta} = c\frac{E (\sum^{\infty} m)}{E (\sum^{\infty} m^{\perp})}$.

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Number of Unordered Samples of Integers With a Given Sum

Tsotne Kutalia

Cybernetics Institute of Georgian Technical University.

Abstract

There is an analytic formula counting the number of ordered samples of N non-negative integers making up a given sum. In this paper we study the number of unordered samples of N non-negative integers with a given sum. We produce a closed form solution for $N = 3$ non-negative integers.

Keywords: Combinatorics, Number Theory, Graph Theory tute p? of

^j §\0 §0²§ §\9§ Tneen° J °0 n@§9_j 9 q}0H_ 0 Tn}h«e9 Tn} N = 3 9jL n N §n @0

$$f_3(n) = I_{\{\{n\}_3=0\}}\left[\frac{(n+3)(n+6)}{18} + I_{\{\{n\}_2=0\}}\frac{n^2}{36} + I_{\{\{n\}_2\neq 0\}}\frac{(n\ \ 3)(n+3)}{36}\right. +$$

$$I_{\{\{n\}_3=1\}}\left[\frac{(n+2)(n+5)}{18} + I_{\{\{n\}_2=0\}}\frac{(n+2)^2}{36} + I_{\{\{n\}_2\neq 0\}}\frac{(n\ \ 1)(n+5)}{36}\right. +$$

$$I_{\{\{n\}_3=2\}}\left[\frac{(n+1)(n+4)}{18} + I_{\{\{n\}_2=0)\}}\frac{(n+4)^2}{36} + I_{\{\{n\}_2\neq 0\}}\frac{(n+1)(n+7)}{36}\right. \qquad \text{r©s}$$

° _H\ }0L«H0 §n

$$f_3(n) = \frac{(n+3\ \ fng_3)(n+6\ \ fng_3)}{18} + \frac{(n+2(fng_3)^2)\ \ (3fng_3)^2}{36} \qquad \text{r¨s}$$

° \0}0 fng_k L0jn§0 9 hnL«en nq0}9§n} X_®_jX 9 }0h9_jL0} Tn} L_®_ _nj nT n n®0} ku

2 Graphical Representation of Partitions, N=3

d0§« L0jn§0 @´ $f_N(n)$ §\0 T«jH§_nj Hn«j§_jX §\0 j«h@0} nT «jn}L0}0L 9hqe0 nT N jnj]j0X9§_®0 _j§0X0} $[a_1; :::; a_N]$ «H\ §\9§ $a_1 + ::: + a_N = n$: d0§ @´ Hnj®0j§_nj $f_0(n) = 1$ Tn} 9ee nu m@®_n« e´ $f_1(n) = 1$ Tn} 9ee n 9 °0eeu ^§ H9j 09 _e´ @0 H\0Hc0L §\9§ Tn} 0®0j n; $f_2(n) = \frac{n+2}{2}$ 9jL Tn} nLL n °0 \9®0 $f_2(n) = \frac{n+1}{2}$u ¯0 H9j §\« L0»j0 $f_2(n)$ °_§\ §\0 _jL_H9§n} T«jH§_nj 9

$$f_2(n) = I_{\{n\ mod\ 2=0\}}\frac{n+2}{2} + I_{\{n\ mod\ 2=1\}}\frac{n+1}{2} \qquad \text{rVs}$$

Sn} N = 3J °0 §9c0 §\0 «h nT N nj0 9jL q9}§_§_nj §\0 «h nT 0}_0 °_§\ © 0q9}9§n} @9} u ¦ _ H9j @0 § @0 _ee« §}9§0L §\}n«X\ 9j 0²9hqe0u Sn} $n = 3$ °0 \9®0 §\0 Tneen°_jX 9}}9jX0h0j§ nT © 0q9}9§n} @9}

$$jj\,1+1+1; \qquad j\,1\,j+1+1; \qquad 1\,j+1\,j+1 \qquad \text{rUs}$$

¦ \0 »} § 9}}9jX0h0j§ _j rUs Hn}}0 qnjL §n $a_1 = 0; a_2 = 0; a_3 = 3$: ¦ \0 0HnjL 9}}9jX0h0j§ Hn}}0 qnjL §n $a_1 = 0; a_2 = 1; a_3 = 2$ 9jL §\0 e9 § nj0 §n $a_1 = 1; a_2 = 1; a_3 = 1$ 2 ^§

§9jL Tn} §\0 §°n @9} qe9H0L 9§ §\0 qn _§_nj o 9jL §\« _§ Hn}}0 qnjL §n §\0 »} § q9}§_§_nj _j rUsu ¦\0 0e0 h0j§ 12 Hn}}0 qnjL §n §\0 0HnjL q9}§_§_nj 9jL }0 q0H§_®0e´ 23 _ Tn} §\0 §_}L q9}§_§_nju
¯0 ®_0° §\0 9hqe0 0e0h0j§ 9 §\0 Hnn}L_j9§0 nT qn_j§ nj §\0 H9}§0 _9j Hnn}L_j9§0 ´ §0h 9jL Tn} Hnj®0j_0jH0 °0 }0®0} 0 §\0 j«h@0} u ~n $f11; 12; 23g$ @0Hnh0 $f11; 21; 32g$u ¦\0 qn_j§ nj §\0 Hnn}L_j9§0 ´ §0h Hn}}0 qnjL_jX §n §_ 9hqe0 _

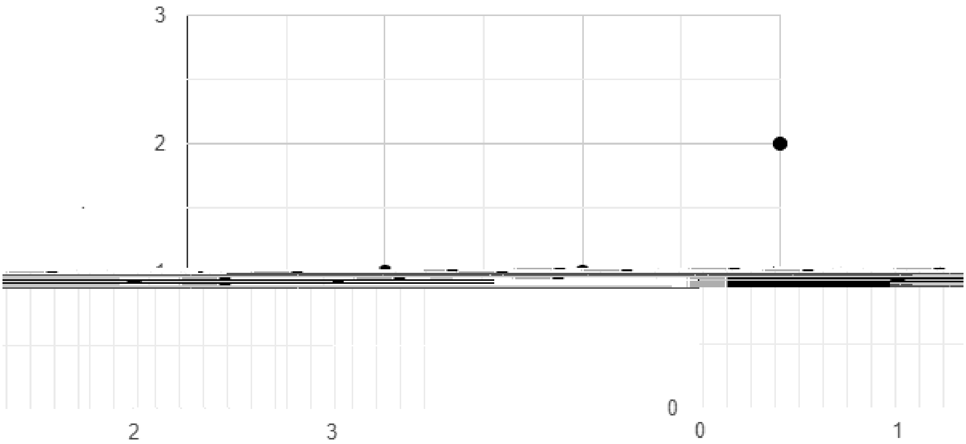

Fig. 1l $f_3(3) = 3$

d_c0°_ 0J Tn} n g

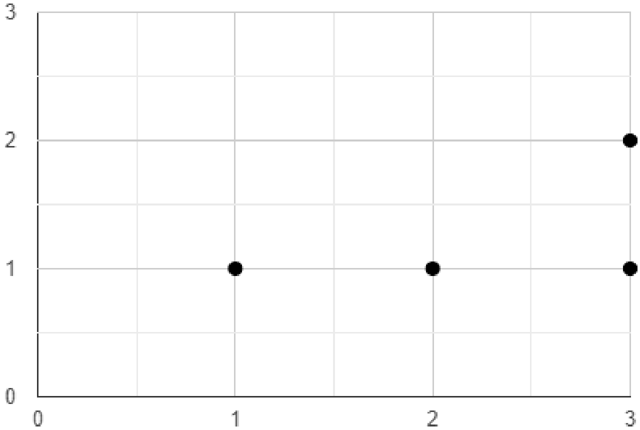

Fig. 2I $f_3(4) = 4$

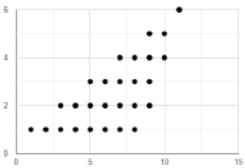

Fig. 3I $f_3(15) = 27$

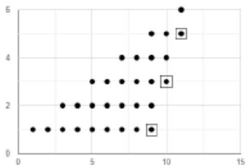

Fig. 4I $f_3(16) = 30$

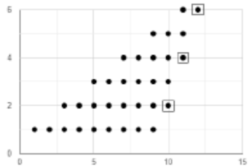

Fig. 5I $f_3(17) = 33$

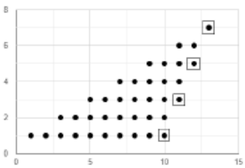

Fig. 6I $f_3(18) = 37$

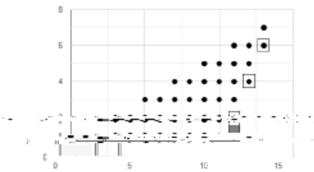

Fig. 7I $f_3(19) = 40$

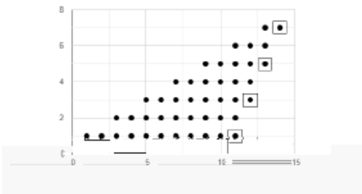

Fig. 8I $f_3(20) = 44$

Tn}h«e9 r©su ¦\0 0²9hqe0 Tn} $N = 3$ 9}0 $n = 15; n = 16; n = 17; n = 18; n = 19; n = 20$u

d0§« @0X_j °_§\ $n = 15; n = 16$ 9jL $n = 17$ nj §\0 nj0 \9jL 9jL $n = 18; n =$

$19; n = 20$ nj 9jn§\0}u ¦\0 }0 q0H§_®0 X}9q\ 9}0 X_®0j _j S_Xu ¨ §n S_Xu Pu ¦\0}0 9}0 nh0 _j§0}0 §_jX q9§§0} 0h0}X_jXu ^j q9}§_H«e9}J °0 \9®0 ¨ qn _@e0 Hnj»X«}9§_nj e_ §0L @0en°

Gnj»X«}9§_nj oJ $n \bmod 3 = 0$: S_Xu¨ L_ qe9´ §\0 H9 0 °\0j $n = 15$ °_H\ _ L_®_ _@e0 @´ ¨u mj §\9§ X}9q\ §\0}0 _ 9j 0²§}0h0 qn_j§ qe9H0L 9§ §\0 Hnn}L_j9§0 $(11; 6)$u ¦_ qn_j§ _ «j_x«0 _j §\0 0j 0 §\9§ _§ Ln0 jn§ \9}0 0_§\0} x n} y Hnn}L_j9§0 °_§\ 9j´ n§\0} qn_j§u ^j X0j0}9eJ §\0}0 _ 9 qn_j§ enH9§0L 9§ §\0 Hnn}L_j9§0 $(x_0; y_0)$ °_e0 §\0}0 _ jn 9j´ n§\0} qn_j§ \9®_jX 0_§\0} x_0 9 x Hnn}L_j9§0 n} y_0 9 y Hnn}L_j9§0u ¦\0 ®9e«0 nT y_0 Hnn}L_j9§0 H9j @0 Tn«jL @´

$$y_0 = 1 + \frac{n}{3}$$ r¢s

Gnj»X«}9§_nj ©J $n \bmod 3 = 1$I S_Xu V L_ qe9´ §\0 H9 0 °\0j $n = 16$u ¦\0 qn_j§ q«§ _j x«9}0 _jL_H9§0 §\0 9LL_§_nj §n §\0 q}0®_n« X}9q\u ~n 9 °0 hn®0 T}nh S_Xu ¨ §n S_Xu V °0 \9®0 j0° qn_j§ 9LL0L nj §\0 Hnn}L_j9§0 $(11; 5)$J $(10; 3)$ 9jL $(9; 1)$u ^j X0j0}9eJ °0 \9®0 §\0 qn_j§ 9LL0L nj §\0 Hnn}L_j9§0 $(x_0; y_0 - 1)$J $(x_0 - 1; y_0 - 3)$ 9jL nnj §_ee §\0 e9 § y Hnn}L_j9§0 }09H\0 0u _u0u $y = 1$u ¦\0 ®9e«0 nT y_0 Hnn}L_j9§0 jn° _

$$y_0 = 1 + \frac{n - 1}{3}$$ rPs

Gnj»X«}9§_nj ¨J $n \bmod 3 = 2$I S_Xu U L_ qe9´ §\0 H9 0 °\0j $n = 17$u 8X9_jJ §\0 qn_j§ _j §\0 x«9}0 _jL_H9§0 §\0 9LL_§_nj T}nh §\0 q}0®_n« H9 0u ^j q9}§_H«e9} °\0j hn®_jX T}nh S_Xu V §n S_Xu U °0 \9®0 §\0 j0° qn_j§ 9LL0L nj §\0 Hnn}L_j9§0 $(12; 6)$J $(11; 4)$ 9jL $(10; 2)$u ^j X0j0}9eJ §\0 qn_j§ 9}0 9LL0L nj $(x_0 + 1; y_0)$J $(x_0; y_0 - 2)$ 9jL nnj §_ee §\0 e9 § qn_j§ y HnnL_j9§0 }09H\0 ©u ¦\0 ®9e«0 nT y_0 Tn} §_ Hnj»X«}9§_nj _

$$y_0 = 1 + \frac{n - 2}{3}$$ rks

^j §n§9e °0 nje´ \9®0 §\0 0 ¨ Hnj»X«}9§_nj 9jL §\0 H´He0 Xn0 n®0} 9jL n®0} 9X9_ju Sn} 0²9hqe0J °\0j $n = 18$J §\0 Hnj»X«}9§_nj ´ §\0 §\0 q v

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Gnj»X«}9§_nj oI ¯0 H9j qe_§ §\0 §n§9e j«h@0} nT qn_j§ nj S_Xu ¨ _j§n §°n q9}§ u ¦ \0 «qq0} q9}§ nT r9jL _jHe«L_jXs §\0 h9_j L_9Xnj9e 9jL §\0 en°0} q9}§u

Fig. 9I $f_3(15) = 27$

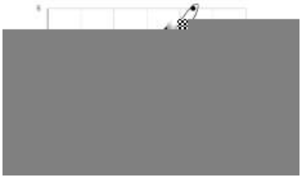

Fig. 10I $f_3(16) = 30$

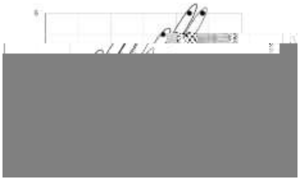

Fig. 11I $f_3(17) = 33$

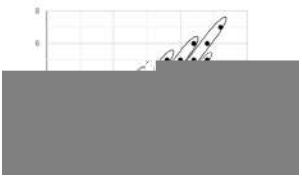

Fig. 12I $f_3(18) = 37$

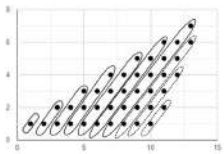

Fig. 13I $f_3(19) = 40$

Fig. 14I $f_3(20) = 44$

¯0 }0T0} §n §\0 Tn}h«e9 nT §\0 «h n §0}h nT 9}_§\h0§_H 0}_0 °_H\ _j _§ hn}0 Hnj®0j_0j§ Tn}h H9j 9HHn}L_jX §n E|NSN|N i GN [N|NF @0 °}_§§0j 9

$$S_n = \frac{n}{2}(a_1 + a_n) \qquad \text{ro·s}$$

°\0}0 a_1 9jL a_n 9}0 }0 q0H§_®0e´ §\0 »} § 9jL §\0 e9 § §0}h nT §\0 0}_0 u ^j Hnj»X«}9§_nj oJ §\0 «qq0} q9}§ nT r9jL _jHe«L_jXs §\0 enjX0 § L_9Xnj9e _ «h h0L 9 $1 + 2 + ::: + (1 + \frac{n}{3})$ °\0}0 §\0 e9 § §0}h Hnh0 T}nh r¢su ?´ ro·s §_ «h _ $\frac{(n+3)(n+6)}{18}$u 8 Tn} §\0 en°0} q9}§ nT §\0 L_9Xnj9eJ °0 \9®0 © ®9}_9§_nj u ^j q9}§_H«e9}J °\0j n _ nLL r§\0 H9 0 \n°j nj S_Xu ks §\0 «h nT §\0 9}_§\h0§_H 0}_0 °_§\ §\0 Hnh hnj L_¸0}0jH0 nT © Hnj _ §_jX nT §\0 Tneen°_jX §0}h $1 + 3 + 5 + ::: + (\frac{n}{3} \quad 1)$ °_H\ @´ ro·s _ $\frac{n^2}{36}$u mj §\0 n§\0} \9jLJ _T n _ 0®0j r§\0 H9 0 \n°j nj S_Xu o©sJ §\0 «h nT §\0 9}_§\h0§_H 0}_0 _ $2 + 4 + 6 + ::: + (\frac{n}{3} \quad 1)$ °_H\ @´ ro·s _ $\frac{(n-3)(n+3)}{36}$u Gnh@_j_jX §\0 0 §0}h ´_0eL §\0 j«h@0} nT «jn}L0}0L 9hqe0 nT 3 jnj] j0X9§_®0 _j§0X0} °_§\ 9 «h n °\0j $n \; mod \; 3 = 0$ °_H\ _

$$I_{\{\{n\}_3=0\}}\left[\frac{(n+3)(n+6)}{18} + I_{\{\{n\}_2=0\}}\frac{n^2}{36} + I_{\{\{n\}_2\neq0\}}\frac{(n \quad 3)(n+3)}{36}\right. : \qquad \text{roos}$$

~_h_e9}e´J Tn} Hnj»X«}9§_nj ©J §\0 «qq0} q9}§ nT r9jL _jHe«L_jXs §\0 enjX0 § L_9Xnj9e _ «h h0L 9 $1 + 2 + ::: + (1 + \frac{n-1}{3})$ °_H\ @´ ro·s _ $\frac{(n+2)(n+5)}{18}$u ¦ \0

$::: + \frac{n-1}{3}$ °_H\ @´ r$\color{blue}{o \cdot}$s _ $\frac{(n-1)(n+5)}{18}$u Sn} n 1 @0_jX 0®0jJ §\0 0}_0 @0Hnh0 $1 + 3 + 5 + ::: + \frac{n-1}{3}$ °_H\ @´ r$\color{blue}{o \cdot}$s _ $\frac{(n+2)^2}{36}$u Gnh@_j_j_X §\0 0 §0}h ´_0eL §\0 j«h@0} nT «jn}L0}0L 9hqe0 nT 3 jnj] j0X9§_®0 _j§0X0} °_§\ 9 «h n °\0j n *mod* 3 = 1 °_H\ _

$$I_{\{\{n\}_3=1\}}\left[\frac{(n+2)(n+5)}{18} + I_{\{\{n\}_2=0\}}\frac{(n+2)^2}{36} + I_{\{\{n\}_2\neq 0\}}\frac{(n \quad 1)(n+5)}{36}\right. :$$

ro©s

S_j9ee´J Tn} Hnj»X«}9§_nj ¨J §\0 «qq0} q9}§ nT r9jL _jHe«L_jXs §\0 enj0 § L_9X] nj9e _ «h h0L 9 $1 + 2 + 3 + ::: + (1 + \frac{n-3}{3})$ °_H\ @´ r$\color{blue}{o \cdot}$s _ $\frac{(n+1)(n+4)}{38}$u ¦\0 en°0} q9}§ _h_e9}e´ §n §\0 q}0®_n« Hnj»X«}9§_nj _ L_¸0} 9HHn}L_jX §n n 2 @0_jX nLL n} 0®0ju Sn} nLL n 2J §\0 «h _ $2 + 4 + ::: + \frac{n-1}{3}$ °_H\ @´ r$\color{blue}{o \cdot}$s _ $\frac{(n+4)^2}{36}$ 9jL Tn} nLL n 2 §\0 «h $1 + 3 + ::: + (1 + \frac{n-3}{3})$ @´ r$\color{blue}{o \cdot}$s _ $\frac{(n+1)(n+7)}{36}$u Gnh@_j_j_X §\0 0 §0}h ´_0eL §\0 j«h@0} nT «jn}L0}0L 9hqe0 nT 3 jnj] j0X9§_®0 _j§0X0} °_§\ 9 «h n °\0j n *mod* 3 = 2 °_H\ _

$$I_{\{\{n\}_3=2\}}\left[\frac{(n+1)(n+4)}{18} + I_{\{\{n\}_2=0\}}\frac{(n+4)^2}{36} + I_{\{\{n\}_2\neq 0\}}\frac{(n+1)(n+7)}{36}\right.$$

ro¨s

^j §n§9eJ $f_3(n)$ §«}j n«§ §n @0 §\0 «h nT r$\color{blue}{oo}$sJ r$\color{blue}{o}$©s 9jL r$\color{blue}{o}$¨s °_H\ _ r$\color{blue}{©}$s }0 §9§0L @0en°

$$f_3(n) = I_{\{\{n\}_3=0\}}\left[\frac{(n+3)(n+6)}{18} + I_{\{\{n\}_2=0\}}\frac{n^2}{36} + I_{\{\{n\}_2\neq 0\}}\frac{(n \quad 3)(n+3)}{36}\right. +$$
$$I_{\{\{n\}_3=1\}}\left[\frac{(n+2)(n+5)}{18} + I_{\{\{n\}_2=0\}}\frac{(n+2)^2}{36} + I_{\{\{n\}_2\neq 0\}}\frac{(n \quad 1)(n+5)}{36}\right. +$$
$$I_{\{\{n\}_3=2\}}\left[\frac{(n+1)(n+4)}{18} + I_{\{\{n\}_2=0)\}}\frac{(n+4)^2}{36} + I_{\{\{n\}_2\neq 0\}}\frac{(n+1)(n+7)}{36}\right.$$

^§ _ 09 _e´ ®0}_»0L §\9§ §\0 Tn}h«e9 9@n®0 H9j @0 }0L«H0L §n r¨s Hnhqnj0j§ @´ Hnhqnj0j§u ¦_ _ 9e n }0 §9§0L @0en°

$$f_3(n) = \frac{(n+3 \quad fng_3)(n+6 \quad fng_3)}{18} + \frac{(n+2(fng_3)^2) \quad (3fng_3)^2}{36}$$

8§ §_ qn_j§ _§ }0h9_j §n q}n®0 §\0 Tn}h«e9u ¦_ _ Lnj0 @´ _jL«H§_nj r$\color{blue}{o}$s q9}§ @´ q9}§u
¦n q}n®0 §\9§ §\0 Hnj»X«}9§_nj o q9}§ nT §\0 Tn}h«e9 \neL Tn} 9j´ n 3J °0 9 «h0 §\9§ _§ \neL Tn} nh0 n 3 9jL \n° §\9§ _§ 9e n \neL Tn} $n + 3$u ^j T9H§J _§ H9j 09 _e´ @0 \n°j §\9§ _T °0 q«§ $n + 3$ _j qe9H0 nT n _j r$\color{blue}{¢}$sJ °0 n@§9_j

§\O Tneen°_jX «h _j qe9HO nT roos

$$
\begin{aligned}
&I_{\{\{n+3\}_3=0\}}(1+2+3+\ldots+(1+\frac{n+3}{3})+\\
&I_{\{\{n+3\}_2=0\}}(1+3+5+\ldots+(\frac{n+3}{3} \quad 1))+ \qquad \text{roVs}\\
&I_{\{\{n+3\}_2\neq 0\}}(2+4+6+\ldots+(\frac{n+3}{3} \quad 1))).
\end{aligned}
$$

8qqe´_jX ro·

3 General Recursive Formula for Arbitrary N and $>= N$

^j §0}h nT hnL«e« nq0}9§n} J ros H9j @0 }0L0»j0L Tn} L_¸0}0j§ N] u Sn} N = 4J °0 \9®0

$$f_4(n) = I_{\{\{n\}_4=0\}} \sum_{k=1}^{\frac{n}{4}+1} f_4$$

^j X0j0}9eJ Tn} 9j 9}@_§}9}´ NJ °0 \9®0 ros

$$f_N(n) = I_{\{\{n\}_N=0\}} \sum_{k=1}^{\frac{n}{N}+1} f_{Nk-N}(N-1) + I_{\{\{n\}_N=1\}} \sum_{k=1}^{\frac{n-1}{N}+1} f_{Nk-N+1}(N-1) + \dots + I_{\{\{n\}_N=N-1\}} \sum_{k=1}^{\frac{n-N+1}{N}+1} f_{Nk-1}(N-1) = \sum_{j=1}^{N-1} I_{\{\{n\}_N=j\}} \sum_{k=1}^{\frac{n-j}{N}+1} f_{Nk-N+j}(N-1)$$

r©¨s

References

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E©F ~_}´90® 8u i uJ N}e_c\ ^uWuJ ³9 cn® pu8uJ p}n@9@_e_§´ _j ¦\0n}0 h 9jL p}n@e0 h J qqu o©

Appendix A Scatter Configurations for $N = 3$

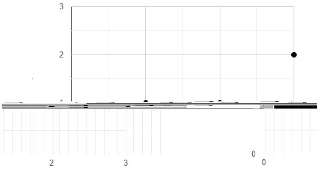


Fig. A1I $f_3(3) = 3$

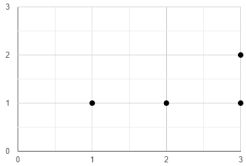


Fig. A2I $f_3(4) = 4$

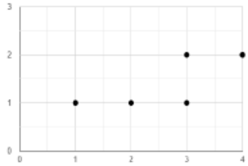


Fig. A3I $f_3(5) = 5$

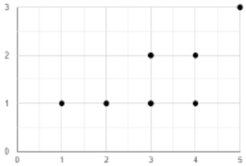


Fig. A4I $f_3(6) = 7$

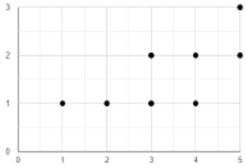


Fig. A5I $f_3(7) = 8$

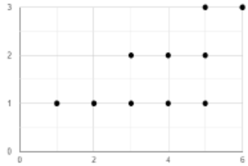


Fig. A6I $f_3(8) = 10$

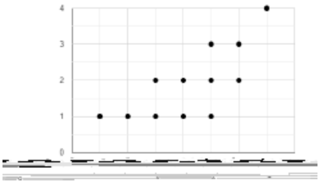


Fig. A7I $f_3(9) = 12$

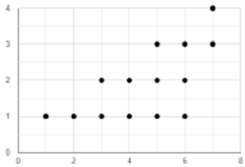


Fig. A8I $f_3(10) = 14$

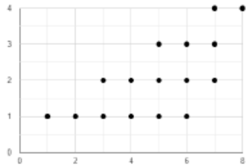


Fig. A9I $f_3(11) = 16$

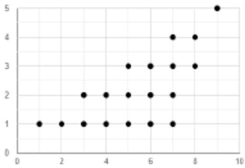


Fig. A10I $f_3(12) = 19$

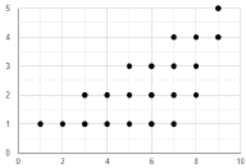


Fig. A11I $f_3(13) = 21$

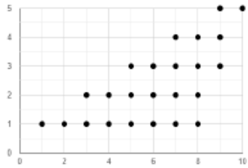


Fig. A12I $f_3(14) = 24$

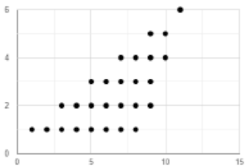


Fig. A13I $f_3(15) = 27$

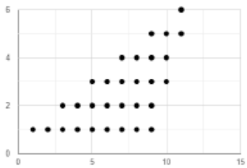


Fig. A14I $f_3(16) = 30$

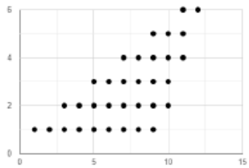
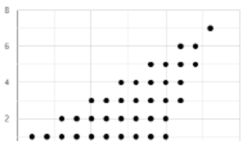
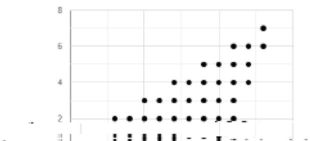
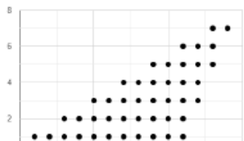


Fig. A15I $f_3(17) = 33$







Construction of identifying and real *M*-estimators in general statistical model with filtration

T. Toronjadze[1,2]

[1]Georgian American University, Business School, 10 Merab Aleksidze Str., 0160, Tbilisi, Georgia;
A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 2 Merab Alekside II Lane, 0193 Tbilisi, Georgia

Abstract

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Key words and phrases: fi ,
M- , b .

MSC 2010: 2 - -1

Let for each *2* and $n \geq 1$ the process (*L* (*; t*)*;* 0

Theorem 1. *Let the following conditions hold:*

a) *for each* $\; 2 \;$, $\lim_{! \, 1} c \; (\;) = 0$;

b) *for each* $n \geq 1$, *the mapping* $\; \rightsquigarrow L \; (\;)$ *is continuously differentiable in* $\; Q_\theta$*-a.s.*, $(L \; (\;) := \frac{\partial}{\partial\theta} L \; (\;))$;

c) *for each* $\; 2 \;$, *there exists a function* $\;_{Q}(\; ; y)$, $\; ; y \, 2 \;$, *such that*

$$Q_\theta\text{-} \lim_{! \, 1} c \; (\;) L \; (y) = \;_{Q}(\; ; y) \tag{4}$$

and the equation

$$\;_{Q}(\; ; y) = 0$$

with respect to the variable y *has the unique solution* $\; = b^{Q}(\;)$;

d) $Q_\theta\text{-} \lim_{! \, 1} c \; (\;) L \; (\; \;) = - \;_{Q}(\;)$, *where* $\;_{Q}(\;)$ *is a positive number for each* $\; 2 \;$;

e) $\lim_{! \, 0} \limsup_{! \, 1} Q_\theta \, f \sup_{f \; : j \; - \theta^* j \leq C} (\;) \ldots$

(ii) $c^{-1}(\ )(T\ -\ \ ) = \dfrac{c^{-1}(\ )L\ (\ \ )}{_Q(\ )} + R\ (\ );\quad R\ (\ ) \xrightarrow{Q_\theta^n} 0.$

Proof. 1. By the Taylor formula we have

$$L\ (y) = L\ (\ \ ) + L\ (\ \ )(y - \ \ ) + [L\ (\ ) - L\ (\ \ )](y - \ \ );$$

where $\ = \ + \ (\ )(y - \ ),\ \ (\ ) 2\ [0;1]$ and the point is chosen so that $2\ F\ \ (\ 2\ F$ means that r.v. is F-measurable).

From this we get

$$c\ (\ )L\ (y) = c\ (\ )L\ (\ \ ) - \ _Q(\ )(y - \ \ ) + "\ (\ ;\ \ )(y - \ \ ); \qquad (5)$$

where $"\ (y;\ \ )\ 2\ F\ $,

$$"\ (y;\ \ ) = c\ (\ )[L\ (y) - L\ (\ \ )] + [c\ (\ )L\ (\ \ ) + \ _Q(\ )];\quad y\ 2\ \ :$$

Evidently, conditions d) and e) ensure that

$$\lim_{\ !\,0} \limsup_{!\,1} Q_\theta \qquad \sup_{!\,1} Q_\theta$$

Obviously, $_\theta(n, r) \in F$. Hence, if $\omega \in {}_\theta(r, n)$, then from equality (7) we get $L(\ + u)u < 0$ for $|u| = r$.

Since the mapping $u \rightsquigarrow L(\ + u)$ is continuous with respect to u, the equation $L(\ + u) = 0$ for $|u| \leq r$ has at least one solution $u(\ )$ with $|u(\ )| \leq r$.

It can be easily seen that if $\omega \in {}_\theta(n, r)$ and $|u| \leq r$, then $L(\ + u) < 0$.

On the other hand, for $\omega \in {}_\theta(n, r)$ and $|u| \leq r$,

$$L(\ + u, \omega) - L(\ + u(\ ), \omega)$$
$$= \int_0^1 \frac{\partial}{\partial}[L((\ + u(\ )) + (u - u(\ )), \omega)]\, d\ .$$

Consequently,

$$L(\ + u, \omega) = \int_0^1 L(\ + u(\ ) + (u - u(\ )), \omega)(u - u(\ ))\, d$$

and

$$L(\ + u, \omega)(u - u(\ ))$$
$$= \int_0^1 L(\ + u(\ ) + (u - u(\ \ L(\text{3.u3.u3.u3.u3.u3.u} \ L(\ )$$

It is easily seen that, by construction, T possesses properties I, II and III.

4. Finally, we prove assertion IV. By expansion (5), we have

$$
\begin{aligned}
&jc\ (\ )L\ (T\ ) - c\ (\ )L\ (\ \ ) - \ _Q(\ )c^{-1}(\ )(T\ -\ \ )j \\
&\quad \leq j''\ (T;\ \ )\ _Q^{-1}(\ )jj\ _Q(\ )c^{-1}(\ )(T\ -\ \ )j
\end{aligned} \tag{8}
$$

and $\limsup_{!\ 1} Q_\theta fj''\ (T\ ;\ \ )j \geq\ \ g = 0$, $8\ \ > 0$, which follows directly from the relation

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$$
fjT\ -\ \ j \leq rg \setminus \sup_{f\ :j\ -\theta^* j \leq,\ g} j''\ (y;\ \ )j < \ \ , \quad fj''\ (T\ ;\ \ )j < \ \ g:
$$

Denote $X\ := c\ (\ )(L\ (T\ ) - L\ (\ \ ))$, $Y\ :=\ _Q(\ )c^{-1}(\ )(T\ -\ \ )$ and $Z\ := j''\ (T\ ;\ \ )\ _Q^{-1}j$. Then inequality (8) takes the form

$$
jX\ - Y\ j \leq Z\ jY\ j:
$$

g

It is well-known that if X converges weakly to X (X

$(\sup c)_1$ *the function* $_Q($

On the other hand,

$$Q_\theta\text{-}\lim_{!\ 1} c\ (\ )L\ (\widehat{T}\ ) = 0$$

and hence,

$$Q_\theta\text{-}\lim_{!\ 1} \quad _Q(\ ;\widehat{T}\ ) = 0: \tag{11}$$

Assume now that equality (9) fails too. Then one can choose " > 0 such that

$$\overline{\lim_{!\ 1}}\ Q_\theta\{$$

Evidently, for any , we have $^{\theta}$ A , where the set $^{\theta}$ is de ned in item 3 of the proof of Theorem 1.

Since under the conditions of Theorem 1, $Q_{\theta} f$ $^{\theta} g$! 1, for any we have

$$\lim_{! \ 1} Q_{\theta} f A \ g = 1:$$

For each $n \geq 1$, introduce the sets:

S = $f\widetilde{T}$: $\widetilde{T}$ is F -measurable; L

The rst and the second terms on the right-hand side converge to zero by virtue of equalities (13) and (14). □

Remark 2. If the conditions of Corollary 1 are satis ed, then by virtue of Theorem 1, IV (ii), there exists an estimator $T = fT\ g_{\ge 1}$ such that

$$T \ = \ + \frac{L\ (\ )}{_Q(\ )} + R\ (\ ); \qquad (15)$$

$$c^{-1}(\ )R\ (\ ) \xrightarrow{Q_\theta^n} 0:$$

If $= b^Q(\ ) =$ and the distribution Φ from Theorem 1, f), is Gaussian, then we obtain a consistent, linear, asymptotically normal estimator.

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ON A GENERALIZATION OF KHINCHIN'S THEOREM

V. BERIKASHVILI, G. GIORGOBIANI, V. KVARATSKHELIA

Abstract. A generalization of Khinchin's theorem for weakly correlated random elements with values in Banach spaces $l_p, 1 \leq p < \infty$ is presented without proof.

[illegible] X *a weak p-order*, $p > 0$, [illegible] $\mathbb{E}|\langle x^*, \xi\rangle|^p < \infty$ [illegible] $x^* \in X^*$. [illegible] *p*- [illegible], $p > 1$, [illegible] *expectation* $\mathbb{E}$ [illegible] *Pettis integral*

Lemma 2.

Corollary 6. *Let $\xi_1, \xi_2, \dots, \xi_n, \dots$ be a sequence of weak second order random elements with values in $l_p, 1 \le p \le 2$, and let the covariance operators $R_n \equiv R_{\xi_n}$ satisfy the condition*

$$\rho_n \equiv \sum_{k=1}^{\infty} \langle e_k, R_n e_k \rangle^{p/2} < \infty, \quad n = 1, 2, \dots.$$

If

$$\frac{1}{n}\left(\sum_{i=1}^{n} \rho_i^p\right)^{2/p} \to 0, \quad n \to \infty \qquad (\;)$$

then the sequence $\xi_1, \xi_2, \dots, \xi_n, \dots$ satisfies the LLN.

$\sum_{i=0}^{n-1} g(i) \le 1$

Corollary 7. *Let $\xi_1, \xi_2, \dots, \xi_n, \dots$ be a sequence of pairwise independent weak second order random elements with values in $l_p, 1 \le p < \infty$, and let for any positive integer n covariance operators $R_n \equiv R_{\xi_n}$ satisfy (3.2).*

If

$$\frac{1}{n^2}\left(\sum_{i=1}^{n} \rho_i^s\right)^{2/s} \to 0, \quad \text{where } s = \max\{2, p\},$$

then the sequence $\xi_1, \xi_2, \dots, \xi_n, \dots$ satisfies the LLN.

Corollary 8. *Let $\xi_1, \xi_2, \dots, \xi_n, \dots$ be a sequence of pairwise independent strong second order random elements with values in a separable Hilbert space and let*

$$\frac{1}{n^2}\sum_{i=1}^{n} tr(R_i) \to 0, \quad n \to \infty.$$

Then the sequence $\xi_1, \xi_2, \dots, \xi_n, \dots$ satisfies the LLN.

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V. Berikashvili, Muskhelishvili Institute of Computational Mathematics, Georgian Technical University, Gr. Peradze str., 4, Tbilisi 0159, Georgia.

E-mail address: .berikash ili@g .ge

G. Giorgobiani, Muskhelishvili Institute of Computational Mathematics, Georgian Technical University, Gr. Peradze str., 4, Tbilisi 0159, Georgia.
E-mail address: giorgobiani.g@g .ge

V. Kvaratskhelia, Muskhelishvili Institute of Computational Mathematics, Georgian Technical University, Gr. Peradze str., 4, Tbilisi 0159, Georgia.
E-mail address: .k ara skhelia@g .ge

Lastly, commonly used option pricing method is Monte Carlo simulation where numerous random paths for the price of an underlying asset are generated, each having an associated payoff. Then present value of payoffs is computed, and their average becomes an option price that values in all simulated scenarios. Just like Binomial Option Pricing model, this method can incorporate any option payoff and dynamically introduced inputs. Moreover, this method is not restricted to a single or any distribution of underlying asset unlike models with closed-form solutions as given by the Black-Scholes. All that gives Monte Carlo simulation substantial number of use-case in real-life applications. Main disadvantage of the method stays to be heavy computational load as it requires a large number of simulations to improve average accuracy.

Next section in this paper introduces more recent approach to option pricing using artificial intelligence, mainly, machine learning (ML) methods. Using same or more number of inputs as in classical options pricing methodologies ML methods can be trained from both simulated and historical data to "learn" either observed option price or theoretical one given by option pricing method of our choice. Success of ML model will depend on quality of training data and its properties for generalization among other things. In case of creating successful ML model that accurately predicts option prices on out-of-sample data, one can conclude that disadvantages of classical option

Above figure suggests that ML models can be used to price options, and in this section similar experiment is carried out now using actual data instead of simulated one for illustrative purposes.

Using daily prices for Nasdaq futures starting from earliest available date as of 19 Sep 2001, at-the-money call and put prices are computed using following input parameters:

- annual standard deviation of continuously compounded returns as a volatility input,
- annual risk-free rate of 3%, and
- time to expiration of 20 trading days.

Code snippet below shows functions used to compute option prices for Nasdaq Futures data till most recent date as of time of writing, 23 Dec 2022.

Code Snippet 1 The Black-Scholes call and put prices, source https://www.codearmo.com/python-tutorial/options-trading-black-scholes-model

```
# functions for option prices

    S  K  T  r  sigma
    d1  np log S K  r  sigma 2 2 T ... sigma np sqrt T
    d2  d1  sigma  np sqrt T
    return S  N d1  K  np exp  r T  N d2

def BS_PUT(S, K, T, r, sigma):
    d1 = (np.log(S/K) + (r + sigma**2/2)*T) / (sigma*np.sqrt(T))
    d2 = d1 - sigma* np.sqrt(T)
    return K*np.exp(-r*T)*N(-d2) - S*N(-d1)
```

Figure below shows complete dataset used for training and testing of ML models; this includes historical data of Nasdaq Futures as well as calculated option prices using formulas in Code Snippet 1.

Figure 3 Nasdaq futures historical prices and option prices by The Black-Scholes (BS) model

Despite high volatility in times series, most recent 5% of complete data was selected as out-of-sample for testing purposes. Out of most common and fundamentally different V.[(fu)4.004(n)2r10.996(m)-3.t3 Td( ) TjETQ0

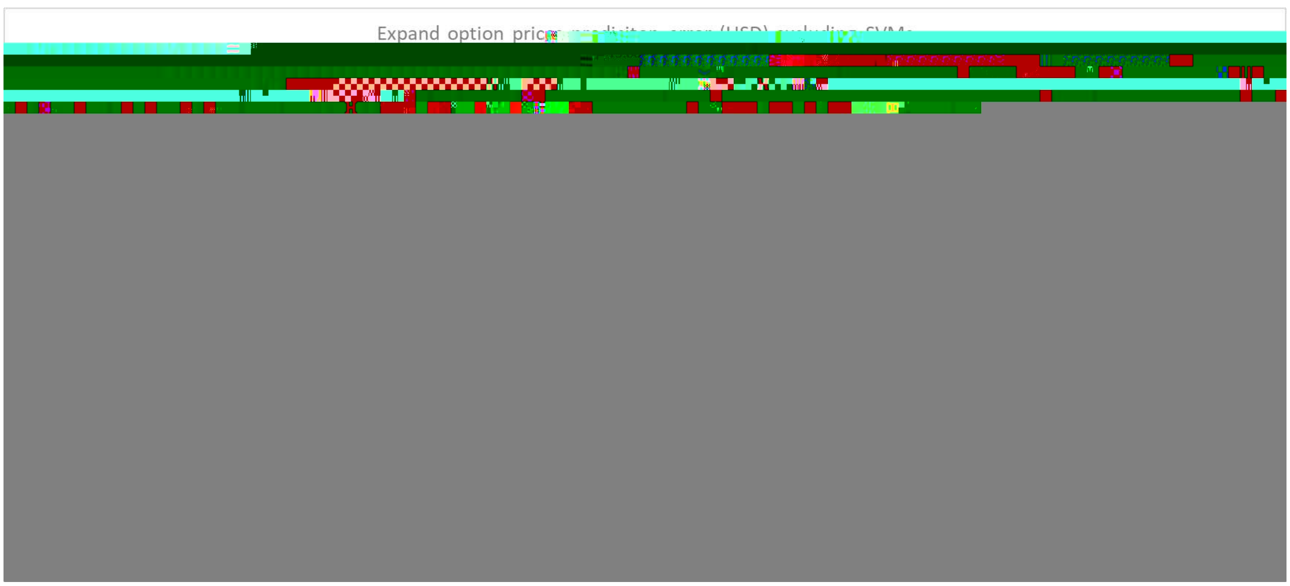
- *gamma*: starting from .01, till 1, with steps of .01

It's worth noting that default configuration for Randomized Grid Search has maximum number of iterations set to 10 and k-fold cross-validation set to 5 folds. Consequently, all models above will get 5-fold cross validation and 10 randomly chosen parameter combinations. Only exception is MLP as it only has 9 possible parameter combinations to search from, in which case exhaustive search will be implemented. It's important to note that there is no right architecture choice for neural networks in general, not so even for

derived by Game Theory models and then biases valuation even further. In this section, option to expand is computed for simulated investment projects data and machine learning models configured in previous section are trained to predict expansion option prices on unseen data. If concept of pricing real options with ML methods is proven, then training ML models

Obviously, SVMs didn't lend themselves to accurately predict real option prices in the example but looking at the graph below excluding SVMs, it's clear that all other ML models seem to predict option prices very closely.

Figure 6 Prediction error for ML models on price of option to expand, excluding SVMs from the mix



Finally, let's look at average error and other prediction metrics in the table below.

Table 2 Options to expand pricing, prediction errors

metric	knn_expand	mlp_expand	lgb_expand	svr_expand
average error	3.09	1.93	0.26	24.03
min error	0.00	0.00	0.00	0.00
max error	16.36	8.80	2.45	131.65
max neg error	-16.36	-8.80	-2.45	-131.65
max pos error	12.81	6.12	1.48	130.13

References

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